

Randomly coloring simple hypergraphs with fewer colors

Alan Frieze* Michael Anastos
 Department of Mathematical Sciences,
 Carnegie Mellon University,
 Pittsburgh PA15213.

Email: alan@random.math.cmu.edu; manastos@andrew.cmu.edu

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Abstract

We study the problem of constructing a (near) uniform random proper q -coloring of a simple k -uniform hypergraph with n vertices and maximum degree Δ . (Proper in that no edge is mono-colored and simple in that two edges have maximum intersection of size one). We show that if $q \geq \max \{C_k \log n, 500k^3 \Delta^{1/(k-1)}\}$ then the Glauber Dynamics will become close to uniform in $O(n \log n)$ time, given a random (improper) start. This improves on the results in Frieze and Melsted [5].

1 Introduction

Markov Chain Monte Carlo (MCMC) is an important tool in sampling from complex distributions. It has been successfully applied in several areas of Computer Science, most notably for estimating the volume of a convex body [5], [12], [13], [3] and estimating the permanent of a non-negative matrix [11].

Generating a (nearly) random q -coloring of a n -vertex graph $G = (V, E)$ with maximum degree Δ is a well-studied problem in Combinatorics [2] and Statistical Physics [14]. Jerrum [10] proved that a simple, popular Markov chain, known as the *Glauber dynamics*, converges to a random q -coloring after $O(n \log n)$ steps, provided $q/\Delta > 2$. This led to the challenging problem of determining the smallest value of q/Δ for which a random q -coloring can be

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generated in time polynomial in n . Vigoda [15] gave the first significant improvement over Jerrum's result, reducing the lower bound on q/Δ to $11/6$ by analyzing a different Markov chain. There has been no success in extending Vigoda's approach to smaller values of q/Δ , and it remains the best bound for general graphs. There are by now several papers giving improvements on [15], but in special cases. See Frieze and Vigoda [8] for a survey.

In this paper we consider the related problem of finding a random coloring of a simple k -uniform hypergraph. A k -uniform hypergraph $H = (V, E)$ has vertex set V and $E = \{e_1, e_2, \dots, e_m\}$ are the edges. Each edge is a k -subset of V . Hypergraph H is simple if $|e_i \cap e_j| \leq 1$ for $i \neq j$. A coloring of H is proper if every edge contains two vertices of a different color. The chromatic number $\chi(H)$ is the smallest number of colors in a proper coloring of H . In the case of graphs $k = 2$ we have $\chi(G) \leq \Delta + 1$ but for hypergraphs ($k \geq 3$) we have much smaller bounds. For example a simple application of the local lemma implies that $\chi(H) = O(\Delta^{1/(k-1)})$. In fact a result of Frieze and Mubayi [7], is that for simple hypergraphs $\chi(H) = O((\Delta/\log \Delta)^{1/(k-1)})$. The proof of [7] is somewhat more involved. It relies on a proof technique called the "nibble". The aim of this note is to show how to improve the results of Frieze and Melsted [6] who showed that under certain circumstances simple hypergraphs can be efficiently randomly colored when there are fewer than Δ colors available. In [6] the number of colors needed was at least n^α for $\alpha = \alpha(k)$, in this paper we reduce the number of colors to polylogarithmic in n .

Large parts of the proofs in [6] are still relevant and so we will quote them in place of reproving them. We have realised that some minor simplifications are possible. So if the proofs in [6] need to be tweaked, we will indicate what is needed in an appendix.

Before formally stating our theorem we will define the Glauber dynamics. All of the aforementioned results on coloring graphs (except Vigoda [15]) analyze the Glauber dynamics, which is a simple and popular Markov chain for generating a random q -coloring.

Let \mathcal{Q} denote the set of proper q -colorings of H . For a coloring $X \in \mathcal{Q}$ we define

$$B(v, X) = \{c \in [q] : \exists e \ni v \text{ such that } X(x) = c \text{ for all } x \in e \setminus \{v\}\}$$

be the set of colors unavailable to v .

Then let $Q = \{1, 2, \dots, q\}$ and

$$A(v, X) = Q \setminus B(v, X).$$

For technical purposes, the state space of the Glauber dynamics is $\Omega = Q^V \supseteq \mathcal{Q}$. From a coloring $X_t \in \Omega$, the evolution $X_t \rightarrow X_{t+1}$ is defined as follows:

Glauber Dynamics

- (a) Choose $v = v(t)$ uniformly at random from V .
- (b) Choose color $c = c(t)$ uniformly at random from $A(v, X_t)$. If $A(v, X_t)$ is empty we let $X_{t+1} = X_t$.

(c) Define X_{t+1} by

$$X_{t+1}(u) = \begin{cases} X_t(u) & u \neq v \\ c & u = v \end{cases}$$

We will assume from now on that

$$q \leq 2\Delta \tag{1}$$

If $q > 2\Delta$ then we defer to Jerrum's result [10].

Let Y denote a coloring chosen uniformly at random from \mathcal{Q} . We will prove the following:

Theorem 1.1. *Let H be a k -uniform simple hypergraph with maximum degree Δ where $k \geq 3$. Suppose that (1) holds and that*

$$q \geq \max \left\{ C_k \log n, 10k\varepsilon^{-1}\Delta^{1/(k-1)} \right\}, \tag{2}$$

where C_k is sufficiently large and depends only on k and

$$\varepsilon = \frac{1}{50k^2}. \tag{3}$$

Suppose that the initial coloring X_0 is chosen randomly from Ω . Then for an arbitrary constant $\delta > 0$ we have

$$d_{TV}(X_t, Y) \leq \delta \tag{4}$$

for $t \geq t_\delta$, where $t_\delta = 2n \log(2n/\delta)$.

Remark 1.2. The definition of ε has been changed from [6]. It results in slightly better bounds for q .

Here d_{TV} denotes variational distance i.e. $\max_{S \subseteq \mathcal{Q}} |\mathbf{Pr}(X_t \in S) - \mathbf{Pr}(Y \in S)|$.

Note that we do not claim rapid mixing from an arbitrary start. Indeed, since we are using relatively few colors, it is possible to choose an initial coloring from which there is no Glauber move i.e. we do not claim that the chain is ergodic, see [6] for examples of blocked colorings.

The algorithm can be used in a standard way, [10], to compute an approximation to the number of proper colorings of H .

We can also prove the following. We can consider Glauber Dynamics as inducing a graph $\Gamma_{\mathcal{Q}}$ on \mathcal{Q} where two colorings are connected by an edge if there is a move taking one to the other. Note that if Glauber can take X to Y in one step, then it can take Y to X in one step.

Corollary 1.3. *The graph $\Gamma_{\mathcal{Q}}$ contains a giant component \mathcal{Q}_0 of size $(1 - o(1))|\mathcal{Q}|$.*

2 Good and bad colorings

Let $X \in \Omega$ be a coloring of V . For a vertex $v \in V$ and $1 \leq i \leq k-1$ let

$$E_{v,i,X} = \{e : v \in e \text{ and } |\{X(w) : w \in e \setminus \{v\}\}| = i\}$$

be the set of edges e containing v in which $e \setminus \{v\}$ uses exactly i distinct colors under X . Let $y_{v,i,X} = |E_{v,i,X}|$ so that $|B(v, X)| \leq y_{v,1,X}$ for all v, X . Let ε be as in (3). We define the sequence $\epsilon = \varepsilon, \varepsilon^2, \dots, \varepsilon^{k-2}$.

Definition 1. We say that X is ϵ -bad if $\exists v \in V, 1 \leq i \leq k-2$ such that

$$y_{v,i,X} \geq \mu_i \text{ where } \mu_i = (\varepsilon q)^i.$$

Otherwise we say that X is ϵ -good.

Remark 2.1. In [6], μ_i is the minimum of $(\varepsilon q)^i$ and a more complicated term. This second term is no longer needed.

Given Definition 1, we have

$$10k\mu_i \leq \mu_{i+1} \leq \varepsilon q \mu_i \quad \text{for } 1 \leq i \leq k-3.$$

It is convenient to define

$$\mu_{k-1} = \Delta.$$

Note that if X is ϵ -good then $|A(v, X)| \geq (1 - \varepsilon)q$ for all $v \in V$.

In this section we will show that almost all colorings of Ω are ϵ -good and almost all colorings in \mathcal{Q} are ϵ -good. This is where we are able to improve our results over [6].

Consider a random coloring $X \in \Omega$. For a vertex $v \in V$ we let $\mathcal{A}_\epsilon(v)$ denote the event $\{v \text{ is not } \epsilon \text{ good}\}$. For an edge $e \in E$ we let \mathcal{B}_e denote the event $\{e \text{ is not properly colored}\}$.

Let \mathbf{Pr}_Ω indicate that the random choice is from Ω and let $\mathbf{Pr}_\mathcal{Q}$ indicate that the random choice is from \mathcal{Q} .

Now consider a dependency graph, in the context of the local lemma. The events are $\mathcal{B}_e, e \in E$ and \mathcal{B}_e and \mathcal{B}_f are independent if $e \cap f = \emptyset$. Note that for $v \in V$ the event \mathcal{A}_v is independent of events not in \mathcal{N}_v , where

$$\mathcal{N}_v = \{f : e \cap f \neq \emptyset \text{ for some } e \ni v\}.$$

Fix an edge $e \in H$. Clearly,

$$p = \mathbf{Pr}_\Omega(\mathcal{B}_e) = \frac{1}{q^{k-1}}.$$

Next let $x_e, e \in E$ satisfy

$$p \leq x_e \prod_{f \in E, f \cap e \neq \emptyset} (1 - x_f). \quad (5)$$

We choose

$$x_e = \theta_q = \frac{2}{q^{k-1}} \leq \frac{1}{2} \text{ for } e \in E.$$

Then, using (2) and $(1 - x) \geq e^{-x/(1-x)}$ for $0 < x < 1$ we see that

$$x_e \prod_{f \in E, f \cap e \neq \emptyset} (1 - x_f) \geq \theta_q \exp \left\{ -\frac{k\Delta\theta_q}{1 - \theta_q} \right\} \geq \theta_q e^{-2k\Delta\theta_q} = \frac{2}{q^{k-1}} \exp \left\{ -\frac{4k\Delta}{q^{k-1}} \right\} \geq \frac{2e^{-1/2}}{q^{k-1}} \geq p.$$

This verifies (5). Theorem 2.1 of Haeupler, Saha and Srinivasan [9] then implies that for $v \in V$ we have

$$\mathbf{Pr}_{\mathcal{Q}}(\mathcal{A}_v) \leq \mathbf{Pr}_{\Omega}(\mathcal{A}_v) \prod_{f \in \mathcal{N}_v} (1 - x_f)^{-1}. \quad (6)$$

This theorem is the basis of our improvement. As stated in [9], there is a short easy proof of this and for completeness we give an outline in an appendix.

Now [6] proves that

$$\mathbf{Pr}_{\Omega}(\mathcal{A}_v) \leq e^{-\varepsilon q}. \quad (7)$$

Remark 2.2. The definition of ε has changed from [6] and so we feel obliged to verify (7) in an appendix.

So, (6) implies that

$$\mathbf{Pr}_{\mathcal{Q}}(\exists v \in V : \mathcal{A}_v) \leq ne^{-\varepsilon q} \prod_{f \in \mathcal{N}_v} (1 - x_f)^{-1} \leq n \exp \left\{ -\varepsilon q + \frac{4k\Delta}{q^{k-1}} \right\} \leq ne^{\frac{1}{2} - \varepsilon q} = o(1),$$

since $q \geq 2\varepsilon^{-1} \log n$.

Thus w.h.p., a q -coloring chosen uniformly at random from either Ω or from \mathcal{Q} is ϵ -good.

3 Persistence of goodness

The following two lemmas are proved in [6]:

Lemma 3.1.

$$\mathbf{Pr}(X_t \text{ is } 2\epsilon\text{-good for } t \leq t_0 \mid X_0 \text{ is } \epsilon\text{-good}) \geq 1 - 2^{-\mu_1/2}.$$

where

$$t_0 = \frac{n}{4k^2e}.$$

Lemma 3.2.

$$\Pr(X_{t_0} \text{ is } \epsilon\text{-good} \mid X_0 \text{ is } \epsilon\text{-good}) \geq 1 - e^{-c\mu_1} \quad \text{for some } c > 0. \quad (8)$$

The constant c in (8) depends only on k . Part of the proof of Lemma 3.2, involves the inequality $\frac{4\epsilon k^2}{1-2\epsilon} \leq \frac{1}{10}$, see (26) of [6]. Our choice of ϵ satisfies the latter inequality.

4 Coupling Argument

Now consider a pair X, Y of copies of our Glauber chain. Let

$$h(X_t, Y_t) = |\{v \in V : X_t(v) \neq Y_t(v)\}|$$

be the Hamming distance between X_t, Y_t . The paper [6] describes a simple coupling between the chains and shows that that

$$\mathbf{E}(h(X_{t+1}, Y_{t+1}) \mid X_t, Y_t) \leq \left(1 - \frac{1}{2n}\right) h(X_t, Y_t)$$

if X_t, Y_t are both 2ϵ -good.

Summarising, we have that with probability at least $1 - e^{-c\mu_1}$ for some positive constant c , we have that both of X_0 and Y_0 are ϵ -good, both X and Y are 2ϵ -good for t_0 steps and both of X_{t_0} and Y_{t_0} are ϵ -good. If we run the chain for $t_0 t^*$ steps, where $t^* = e^{c\mu_1/2}$ then the probability that either chain stops being 2ϵ -good is at most $t^* e^{-c\mu_1} = e^{-c\mu_1/2}$. Conditional on these events, $\mathbf{E}(h(X_{t_\delta}, Y_{t_\delta}) \leq \delta/2$ and together with the fact that the variation distance between X_t and Y_t is monotone non-increasing, this implies (4). (Note that δ includes the probability that either of X_0, Y_0 are not 2ϵ -good). Choosing C_k large enough so that $c\epsilon C_k = \frac{cC_k}{50k^2} \geq 4$ completes the proof of Theorem 1.1. (This choice of C_k ensures that $t^* \geq n^2$.)

4.1 Proof of Corollary 1.3

The proof of Theorem 1.1 shows that if $X, Y \in \mathcal{Q}$ are both ϵ -good then there is a path from X to Y in \mathcal{Q} of length $O(n \log n)$. Since almost all of \mathcal{Q} is ϵ -good, we are done. \square

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A Proof of (6)

$$\begin{aligned}
\Pr_{\mathcal{Q}}(\mathcal{B}_v) &= \Pr_{\Omega}(\mathcal{B}_v \mid \bigcap_{e \in E} \mathcal{B}_e) \\
&= \frac{\Pr_{\Omega}(\mathcal{B}_v \cap \bigcap_{e \in \mathcal{N}_v} \mathcal{B}_e \mid \bigcap_{e \notin \mathcal{N}_v} \mathcal{B}_e)}{\Pr_{\Omega}(\bigcap_{e \in \mathcal{N}_v} \mathcal{B}_e \mid \bigcap_{e \notin \mathcal{N}_v} \mathcal{B}_e)} \\
&\leq \frac{\Pr_{\Omega}(\mathcal{B}_v \mid \bigcap_{e \notin \mathcal{N}_v} \mathcal{B}_e)}{\Pr_{\Omega}(\bigcap_{e \in \mathcal{N}_v} \mathcal{B}_e \mid \bigcap_{e \notin \mathcal{N}_v} \mathcal{B}_e)} \\
&= \frac{\Pr_{\Omega}(\mathcal{B}_v)}{\Pr_{\Omega}(\bigcap_{e \in \mathcal{N}_v} \mathcal{B}_e \mid \bigcap_{e \notin \mathcal{N}_v} \mathcal{B}_e)} \\
&\leq \frac{\Pr_{\Omega}(\mathcal{A}_v)}{\prod_{e \in \mathcal{N}_v} (1 - x_e)}. \tag{9}
\end{aligned}$$

Inequality (9) follows from a standard proof of the Lovász Local Lemma, see for example Alon and Spencer [1], 3rd Edition, (5.4).

B Proof of (7)

The proof in [6] starts with (10):

$$\begin{aligned}
\Pr(y_{v,i,X} \geq \mu_i) &\leq \left(\frac{e \binom{k-1}{i} (i/q)^{k-1-i} \Delta}{\mu_i} \right)^{\mu_i} \\
&= \left(\frac{e \binom{k-1}{i} (i/q)^{k-1-i} \Delta}{(\varepsilon q)^i} \right)^{\mu_i} \\
&\leq \left(\frac{k^i e^{i+1} i^{k-1} \Delta}{i^{2i} (\varepsilon q)^{k-1}} \right)^{\mu_i} \\
&\leq \left(\frac{k^i e^{i+1} i^{k-1}}{i^{2i} (500k^2)^{k-1}} \right)^{\mu_i} \\
&\leq 10^{-2\mu_i} \\
&\leq 10^{-2\varepsilon q}.
\end{aligned} \tag{10}$$